

Quantum mechanics and classical trajectories

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Abstract

The classical limit $\hbar \rightarrow 0$ of quantum mechanics is known to be delicate, in particular there seems to be no simple derivation of the classical Hamilton equation, starting from Schrödinger's equation. In this paper I elaborate on an idea of M. Reuter [8] to represent wave functions by parallel sections of a flat vector bundle over phase space, using the connection of Fedosov's construction of deformation quantization. This generalizes the ordinary Schrödinger representation, and allows naturally for a description of quantum states in terms of a curve plus a wave function. Hamilton's equation arises in this context as a condition on the curve, ensuring the dynamics to split into a classical and a quantum part.

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1 Introduction

In the usual formalism of quantum mechanics, where pure states are time-dependent elements of a Hilbert space on which observables act as linear operators, the dynamics is governed by Schrödinger's equation, whereas in the classical regime $\hbar \rightarrow 0$ pure states are represented by trajectories in phase space obeying Hamilton's equation and observables are functions on phase space. In the classical limit $\hbar \rightarrow 0$ one should therefore be able to associate a curve to a wave function, and to prove that it satisfies Hamilton's equation if the wave function solves Schrödinger's equation.

This problem turns out to be surprisingly hard, and it has found its most satisfying solution in a somewhat alternative formulation of quantum mechanics, in the work of Groenewold and Moyal [3, 6]. They introduced the so-called star product, which allows to treat quantum observables as functions on phase space, like in the classical regime. Only the algebra structure is modified, reflecting the non-commutativity of operators. States can then be defined very formally as functionals on phase space, both classically and quantized, and the time-dependence is shifted to the observables, as in the Heisenberg picture. The evolution equation for an observable f reads

$$\frac{d}{dt}f = \frac{i}{\hbar}(H *_{\hbar} f - f *_{\hbar} H), \quad (1.1)$$

where H is the Hamilton function of the system, and $*_{\hbar}$ the Groenewold-Moyal product for phase space \mathbb{R}^{2n} , or more generally any star-product on a symplectic manifold. In the classical limit $\hbar \rightarrow 0$ the product $*_{\hbar}$ is required to satisfy $(f *_{\hbar} g - g *_{\hbar} f) \rightarrow i\hbar\{f, g\}$ for any two observables f, g , where on the right hand side we have the Poisson bracket of f and g . Thus in the classical limit we obtain

$$\frac{d}{dt}f = \{f, H\}, \quad (1.2)$$

which is equivalent to Hamilton's equation in the usual picture of time-dependent states and stationary observables. Besides solving the classical limit problem, the star-product approach to quantization has proved very useful and led to the concept of deformation quantization, culminating in the work of Fedosov [2] and Kontsevich [5].

In the present note I propose a reformulation of ordinary quantum mechanics, using ideas from deformation quantization as well, that allows us to derive Hamilton's equation directly from the Schrödinger equation. Instead of working with a deformed product we use Fedosov's method to represent observables as parallel sections of a flat vector bundle, and consider a particular representation of the resulting algebra, following ideas of Reuter [8]. In the end our method looks very similar to the textbook formulation of quantum mechanics, and can be formulated without recourse to deformation quantization.

We will treat pure quantum states as equivalence classes $[c, \psi]$, where $c : \mathbb{R} \rightarrow \mathbb{R}^{2n}$ is a curve in phase space and $\psi \in L^2(\mathbb{R}^n)$ is an ordinary wave function. The interpretation is that to every point ξ in our phase space is attached a Hilbert space \mathcal{H}_ξ , all of them isomorphic to $\mathcal{H}_0 = L^2(\mathbb{R}^n)$, and that $\psi(t)$ takes values in $\mathcal{H}_{c(t)}$. In other words we have a trivial Hilbert bundle $\mathcal{H} \rightarrow \mathbb{R}^{2n}$ over phase space, and ψ is a section of the pullback bundle $c^*\mathcal{H}$. The equivalence relation is defined in terms of a parallel transport operator U which is used to identify the different Hilbert spaces: for all points $\xi, \sigma \in \mathbb{R}^{2n}$ we have an isomorphism

$$U(\xi, \sigma) : \mathcal{H}_\xi \rightarrow \mathcal{H}_\sigma, \quad (1.3)$$

and we identify two pairs (c, ψ) and (c', ψ') if and only if the relation $\psi'(t) = U(c(t), c'(t))\psi(t)$ holds for all t .

The question remains how to define the parallel transport U . As observed by Reuter in [8] Fedosov's construction of deformation quantization yields a connection on our Hilbert bundle \mathcal{H} , and this defines the parallel transport, up to a minor modification which is necessary because Fedosov's connection is not flat, and therefore its parallel transport is path-dependent. This will be fixed by tensoring \mathcal{H} with a line-bundle B with connection, in fact the prequantum line-bundle from geometric quantization. The resulting bundle has indeed a flat connection [7]. Instead of using pairs $[c, \psi]$ to describe a pure state we can then equivalently use a time-dependent parallel section Ψ of $\mathcal{H} \otimes B$. Its value at $\xi \in \mathbb{R}^{2n}$ is obtained by parallel-transport of $\psi(t)$:

$$\Psi_\xi(t) = U(c(t), \xi)\psi(t) \quad \in (\mathcal{H} \otimes B)_\xi. \quad (1.4)$$

Similarly, the operator $\rho(f)$ corresponding to a classical observable f can be represented by a parallel section w.r.t. the induced connection on the endomorphism bundle. This is the main idea of Fedosov's construction.

Due to the equivalence relation the choice of the curve c to represent a state $[c, \psi]$ is completely arbitrary: for every pure state η and every curve c we can find a wave function ψ such that $\eta = [c, \psi]$. In particular, choosing c to remain in the origin, $c \equiv 0$, we recover the textbook formulation of quantum mechanics, but for other trajectories we get different (equivalent) representations. The dynamics in this picture is governed by what we call the Schrödinger equation over c . As usual this is a differential equation for ψ , whose explicit form depends on the choice of c , however. If c is constant it looks similar to the usual Schrödinger equation, but for non-constant c an important modification occurs.

As any curve can be used to represent a given quantum state, in combination with an appropriate wave function, we can ask whether there is a suitable notion of preferred trajectories. Taking a closer look at expectation values of observables we find that they naturally split into a classical and a quantum contribution, where the classical part is uniquely determined by the trajectory c alone. Apparently this splitting then depends on the choice of representative for the equivalence class $[c, \psi]$. We will define the preferred trajectories as those where the quantum contribution to expectation values remains small (i.e. of order \hbar) under time evolution. The main result of the paper is that these preferred trajectories are exactly the solutions of Hamilton's equation.

This can be understood as a derivation of Hamilton's equation from the Schrödinger equation. Starting with a quantum state $[c, \psi]$ solving Schrödinger's equation, the measurement of any observable f in this state in the classical limit $\hbar \rightarrow 0$ gives the value $f(c(t))$ if c solves Hamilton's equation with appropriate boundary condition, and the disappearance of the wave function in the

classical limit is not at all mysterious any more. It is a consequence of a good choice of representative for every state.

The derivation of this result requires some assumptions on the initial wave function $\psi(t_0)$, which are summarized in theorem 2.1. Basically one has to make sure that the wave function does not spread over a macroscopic area (neither in position space, nor in momentum space when Fourier transformed). This could have been expected, because even classically the center of mass motion in this case would not be described by a trajectory solving Hamilton's equation, but by some phase space density obeying Liouville's equation. Finally, the generalization to quantizable symplectic manifolds is indicated in section 4.

2 Quantum mechanics over phase space

Deformation quantization. We consider a trivial Hilbert bundle $\mathcal{H} \rightarrow \mathbb{R}^{2n}$ over phase space, i.e. a collection of Hilbert spaces $\mathcal{H}_\xi \simeq L^2(\mathbb{R}^n)$ for every point $\xi \in \mathbb{R}^{2n}$. They come equipped with an action of the Weyl algebra, generated by \hat{q}^j and \hat{p}_k (where $j, k = 1, \dots, n$):

$$\hat{q}^j \psi(x) = x^j \psi(x), \quad \hat{p}_k \psi(x) = \frac{\hbar}{i} \frac{\partial}{\partial x^k} \psi(x)$$

for $\psi \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. We will denote these generators collectively by y^μ ($\mu = 1, \dots, 2n$), they satisfy the canonical commutation relations $[y^\mu, y^\nu] = i\hbar \omega^{\mu\nu}$, where $\omega = dp_j \wedge dq^j$ is the standard symplectic form on \mathbb{R}^{2n} with linear coordinates q^j, p_k ($j, k = 1, \dots, n$). States Ψ will be sections of the Hilbert bundle, so that they depend both on the phase space point ξ as well as an auxiliary variable $x \in \mathbb{R}^n$. Since we introduced additional degrees of freedom this way as compared to the usual formulation of quantum mechanics, we need to impose a further constraint on physical states, and we require them to be parallel:

$$D\Psi = 0, \tag{2.1}$$

where

$$D = d - \frac{i}{\hbar} \left[\theta + \omega_{ab} y^a d\xi^b \right] \tag{2.2}$$

is a connection on our trivial Hilbert bundle, with $d = dq^j \frac{\partial}{\partial q^j} + dp_j \frac{\partial}{\partial p_j}$ being the exterior derivative, and θ any 1-form on \mathbb{R}^{2n} satisfying $d\theta = \omega$. The results will be independent of the precise form of θ , but for the example below a convenient choice turns out to be $\theta = \frac{1}{2}(p_a dq^a - q^a dp_a)$, which we therefore adopt. The solutions of $D\Psi = 0$ have the form

$$\Psi_{(q,p)}(x) = \chi(q+x) e^{-\frac{i}{\hbar} p(x + \frac{q}{2})}, \tag{2.3}$$

where χ is any square-integrable function on \mathbb{R}^n , and for $p, x \in \mathbb{R}^n$ the expression px denotes the standard inner product $p_j x^j$. The operator $\rho(f)$ corresponding to a function f , satisfying $[D, \rho(f)] = 0$ and acting on \mathcal{H}_ξ , is

$$\begin{aligned} \rho(f)_\xi &= \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{\mu_1} \dots \partial_{\mu_k} f(\xi) y^{\mu_1} \dots y^{\mu_k} \\ &= f(\xi) + \partial_\mu f(\xi) y^\mu + \frac{1}{2} \partial_\mu \partial_\nu f(\xi) y^\mu y^\nu + \dots, \end{aligned} \tag{2.4}$$

where $y^{\mu_1} \dots y^{\mu_k}$ denotes the symmetrized product of the operators $y^{\mu_1}, \dots, y^{\mu_k}$ [2]. If we choose the base point equal to the origin, i.e. $\xi = 0$, then (2.4) reproduces the standard quantization prescription in Weyl ordering. Finally, the parallel transport operator

$$U := U((q_0, p_0), (q, p)) : \mathcal{H}_{(q_0, p_0)} \rightarrow \mathcal{H}_{(q, p)}, \quad (2.5)$$

defined by $\Psi_{(q, p)} = U \Psi_{(q_0, p_0)}$ for every physical state Ψ , is a Weyl operator

$$U = \exp \left[\frac{i}{\hbar} \left((p_0 - p) \hat{q} + (q - q_0) \hat{p} + \frac{1}{2} (qp_0 - pq_0) \right) \right]. \quad (2.6)$$

The important property we need is that U satisfies the parallel transport equation

$$\partial_t U(\xi, c(t)) = -A_{c(t)}(\dot{c}(t)) U(\xi, c(t)), \quad (2.7)$$

where $\xi \in \mathbb{R}^{2n}$, c is any curve starting in ξ , and A is the connection 1-form of D , i.e.

$$A = -\frac{i}{\hbar} \left[\theta + \omega_{ab} y^a d\xi^b \right]. \quad (2.8)$$

Dynamics over a fixed base point. Let H be a classical Hamilton function and $\psi \in \mathcal{H}_\xi$ a wave function over $\xi \in \mathbb{R}^{2n}$. Then the Schrödinger equation for ψ is

$$i\hbar \partial_t \psi = \rho(H)_\xi \psi, \quad (2.9)$$

and the expectation value of an observable $f \in C^\infty(\mathbb{R}^{2n})$ is given by (we assume ψ to be normalized)

$$\langle f \rangle_\psi = \langle \psi, \rho(f)_\xi \psi \rangle = f(\xi) + \partial_\mu f(\xi) \langle \psi, y^\mu \psi \rangle + \frac{1}{2} \partial_\mu \partial_\nu f(\xi) \langle \psi, y^\mu y^\nu \psi \rangle + \dots \quad (2.10)$$

The first term gives a classical contribution, i.e. just the value of the function f at ξ , the next terms are quantum corrections. A natural expectation is that the latter ones should be small compared to the classical contribution, but this cannot be true in general, as the classical part is determined by the completely arbitrary choice of base point and remains constant in time. To make this more precise, we will consider \hbar as a formal variable from now on (partly motivated by Fedosov's strategy in [2]). Wave functions will have to be considered as formal Laurent series in some power of \hbar then, whereas expectation values of \hbar -independent observables should become power series in $\hbar^{1/2}$, and we say a term is small or microscopic if it is of order $\hbar^{1/2}$ at least, and large or macroscopic, if it is of order $\hbar^0 = 1$.

The relation $[y^\mu, y^\nu] = i\hbar \omega^{\mu\nu}$ suggests to assign the \hbar -degree $1/2$ to each of the y^μ , and to expect terms $\langle \psi, y^{\mu_1} \dots y^{\mu_k} \psi \rangle$ to be of $\mathcal{O}(\hbar^{k/2})$. This \hbar -grading has turned out to be a very powerful tool in the construction of quantum deformations of general curved phase spaces [2], and it is perhaps surprising to find it not being respected by taking expectation values. To see explicitly why this expectation fails here, let us consider the time evolution of expectation values of the simple coordinate functions q^j and p_k . Let $\xi = (a, b)$, where $a \in \mathbb{R}^n$ are the q -components, and b the p -components, then the splitting into classical and quantum contributions is

$$\langle q^j \rangle_\psi = a^j + \langle \psi, \hat{q}^j \psi \rangle, \quad \langle p_k \rangle_\psi = b_k + \langle \psi, \hat{p}_k \psi \rangle. \quad (2.11)$$

The Schrödinger equation for ψ implies

$$\begin{aligned}
\partial_t \langle \psi, y^\mu \psi \rangle &= \frac{i}{\hbar} \langle \psi, [\rho(H), y^\mu] \psi \rangle \\
&= \frac{i}{\hbar} \left[\partial_\alpha H \langle \psi, [y^\alpha, y^\mu] \psi \rangle + \frac{1}{2} \partial_\alpha \partial_\beta H \langle \psi, [y^\alpha y^\beta, y^\mu] \psi \rangle + \dots \right] \\
&= \underbrace{\omega^{\mu\alpha} \partial_\alpha H}_{\hbar^0} + \underbrace{\omega^{\mu\alpha} \partial_\alpha \partial_\beta H \langle \psi | y^\beta \psi \rangle}_{\hbar^{1/2}} + \underbrace{\dots}_{\mathcal{O}(\hbar^1)}
\end{aligned} \tag{2.12}$$

where the indicated \hbar -degrees are the ones one would naively expect, according to the filtration explained above. The first term on the right hand side is responsible for the fact that under time evolution the quantum part $\langle \psi, \hat{q}^j \psi \rangle$ assumes macroscopic values, even if it was of order $\hbar^{1/2}$ initially. Then by (2.10) this is also true for the quantum corrections of other observables. If the operator $\rho(H) = H + \partial_\mu H y^\mu + \frac{1}{2} \partial_\mu \partial_\nu H y^\mu y^\nu + \dots$ had no terms linear in y^μ , the \hbar -filtration would be preserved under time-evolution.

Dynamics over a curve. Let us then consider the Schrödinger equation over a curve. Suppose that $\psi \in \mathcal{H}_\xi$ is a wave function solving the Schrödinger equation (2.9), and c a curve starting in ξ . Then the wave function $\phi \in \Gamma(c^* \mathcal{H})$, defined by

$$\phi(t) := U(\xi, c(t)) \psi(t) \in \mathcal{H}_{c(t)}$$

satisfies the Schrödinger equation along c :

$$\begin{aligned}
i\hbar \partial_t \phi &= i\hbar (\partial_t U) \psi + i\hbar U \partial_t \psi \\
&= -i\hbar A(\dot{c}) U \psi + U \rho(H)_\xi \psi \\
&= (\rho(H)_{c(t)} - i\hbar A_{c(t)}(\dot{c}(t))) \phi(t),
\end{aligned} \tag{2.13}$$

where we used that $\rho(H)_{c(t)} = U \rho(H)_\xi U^{-1}$. Inserting the explicit expressions for $\rho(H)$ and A we find that

$$i\hbar \partial_t \phi = \left(H - \theta(\dot{c}) + (\partial_\mu H - \omega_{\mu\nu} \dot{c}^\nu) y^\mu + \frac{1}{2} \partial_\mu \partial_\nu H y^\mu y^\nu + \dots \right) \Big|_{c(t)} \phi(t) \tag{2.14}$$

Only the terms of \hbar -degree 0 and 1/2 have changed compared to the usual Schrödinger equation (2.9), and the degree 1/2 term vanishes if and only if c satisfies Hamilton's equation

$$\partial_t c^\mu(t) = \omega^{\mu\nu} \partial_\nu H(c(t)). \tag{2.15}$$

In this case we get

$$i\hbar \partial_t \phi = \left(H - \theta_\mu \omega^{\mu\nu} \partial_\nu H + \sum_{k=2}^{\infty} \frac{1}{k!} \partial_{\mu_1} \dots \partial_{\mu_k} H y^{\mu_1} \dots y^{\mu_k} \right) \Big|_{c(t)} \phi, \tag{2.16}$$

and the \hbar -filtration is preserved under taking commutators with the modified Hamilton operator on the right hand side. The time evolution of the lowest quantum correction $\langle \phi, y^\mu \phi \rangle$ becomes

$$\partial_t \langle \phi, y^\mu \phi \rangle = \omega^{\mu\alpha} \partial_\alpha \partial_\beta H \langle \phi, y^\beta \phi \rangle + \mathcal{O}(\hbar^1), \tag{2.17}$$

and similarly for the expectation values of higher powers of the canonical operators. Therefore, if initially the \hbar -filtration is respected by the expectation values, so that $\langle \phi(t_0), y^{\mu_1} \dots y^{\mu_k} \phi(t_0) \rangle$ is of

order $\hbar^{k/2}$, then this filtration will be preserved under time-evolution. Accordingly, the quantum corrections to the classical result $f(c(t))$ of the expectation value

$$\langle f \rangle_\phi = f(c(t)) + \partial_\mu f(c(t)) \langle \phi, y^\mu \phi \rangle + \frac{1}{2} \partial_\mu \partial_\nu f(c(t)) \langle \phi, y^\mu y^\nu \phi \rangle + \dots \quad (2.18)$$

remain of order $\hbar^{1/2}$ and vanish in the classical limit $\hbar \rightarrow 0$. In other words, we have deduced Hamilton's equation from the Schrödinger equation by taking the classical limit. To collect the assumptions, we formulate the result as a theorem:

Theorem 2.1. *Let $\phi(t) \in L^2(\mathbb{R}^n)$ satisfy the Schrödinger equation (2.13) over a curve c solving Hamilton's equation (2.15). Assume that at some initial time t_0 the expectation values of the canonical operators y^μ satisfy*

$$\langle \phi(t_0), y^{\mu_1} \dots y^{\mu_k} \phi(t_0) \rangle = \mathcal{O}(\hbar^{k/2}) \quad (2.19)$$

for all $k \geq 1$ and $\mu_j = 1, \dots, 2n$. Then the classical limit of the expectation value $\langle f \rangle_{\phi(t)}$, for an observable $f \in C^\infty(M)$, is

$$\lim_{\hbar \rightarrow 0} \langle f \rangle_{\phi(t)} = f(c(t)).$$

The limit $\hbar \rightarrow 0$ means of course to discard all terms of positive \hbar -degree. The assumptions on the expectation values are satisfied e.g. by the eigenfunctions of the harmonic oscillator, and also by coherent states. For the latter ones the theorem has been proven in [4] already.

What is the physical meaning of these conditions? If e.g. $\langle \hat{q}^2 \rangle$ is of order $\hbar^0 = 1$, then the wave function spreads over a macroscopic area, and even classically one does not expect the center of mass motion to coincide with that of a localized point particle. It is then tempting to ask whether there is a density ρ on phase space which under time evolution w.r.t. the Liouville equation gives rise to the same expectation values as ϕ , up to quantum corrections, even without the localization condition. Here we shall not pursue this question further, but content us with the explanation of the assumption in the theorem.

3 Example: harmonic oscillator

Consider the Hamiltonian

$$H(q, p) = \frac{1}{2}(p^2 + q^2). \quad (3.1)$$

on a two-dimensional phase space. Its corresponding quantum operator is

$$\rho(H)_{(q,p)} = \frac{1}{2}(q^2 + p^2) + q\hat{q} + p\hat{p} + \frac{1}{2}(\hat{q}^2 + \hat{p}^2). \quad (3.2)$$

Classical solutions $c : \mathbb{R} \rightarrow \mathbb{R}^2$ of Hamilton's equation

$$\partial_t c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} c$$

are of the form

$$c(t) = \exp \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t \right\} c(0) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} c(0). \quad (3.3)$$

Then the Schrödinger equation along c , equation (2.16) with $\theta = \frac{1}{2}(pdq - qdp)$, reads

$$i\hbar\partial_t\phi = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)\phi, \quad (3.4)$$

i.e. the ordinary oscillator equation, although we are working over the curve c now. So if ϕ initially is an eigenstate $|n\rangle$ of $\frac{1}{2}(\hat{p}^2 + \hat{q}^2)$, it remains so and describes quantum fluctuations around the motion of a particle moving along the classical trajectory (3.3). As H contains no monomials of degree higher than 2 in q and p , and $\langle n|y^\mu n\rangle = 0$, equation (2.11) implies that

$$\langle q\rangle_\phi = c^q(t) = \cos(t)c^q(0) + \sin(t)c^p(0), \quad (3.5)$$

and there are no quantum corrections at all to the center of mass motion. What do these states look like in the standard Hilbert space \mathcal{H}_0 ? Let's take $\phi(0)$ as the ground state $|0\rangle$, then $\phi(t) = e^{-\frac{it}{2}}|0\rangle$, and the corresponding wave function of textbook quantum mechanics, $\psi \in \mathcal{H}_0$, is given by

$$\psi(t) = U(c(t), 0)\phi(t) = \exp\left[\frac{i}{\hbar}\left(c^p(t)\hat{q} - c^q(t)\hat{p}\right)\right]\phi(t). \quad (3.6)$$

In terms of the creation and annihilation operators

$$\hat{a} = \sqrt{\frac{1}{2\hbar}}(\hat{q} + i\hat{p}), \quad \hat{a}^\dagger = \sqrt{\frac{1}{2\hbar}}(\hat{q} - i\hat{p})$$

this reads

$$\psi(t) = e^{-\frac{it}{2}} \exp\left(ze^{-it}\hat{a}^\dagger - \bar{z}e^{it}\hat{a}\right)|0\rangle, \quad (3.7)$$

with $z = c^q(0) + ic^p(0)$. These are the 'coherent states', which resemble classical solutions as closely as possible.

4 Curved phase spaces

Theorem 2.1 generalizes to the case of an arbitrary symplectic phase space (M, ω) satisfying the usual quantization conditions that $\frac{[\omega]}{2\pi\hbar} \in H^2(M, \mathbb{Z})$ and $c_1(M)$ is even. In this case there is again a Hilbert bundle $\mathcal{H} \rightarrow M$, the bundle of 'metaplectic spinors', and one can find a so called prequantum line bundle $B \rightarrow M$. On \mathcal{H} one has Fedosov's connection at one's disposal, of curvature $\frac{i}{\hbar}\omega$, and B carries a connection of curvature $-\frac{i}{\hbar}\omega$. The tensor product $\mathcal{H} \otimes B$ thus has a flat connection, and the wave functions are parallel sections of $\mathcal{H} \otimes B$. Furthermore there is an action of the observables $C^\infty(M)$ on the space of wave functions. At least on cotangent bundles one can prove that this representation of quantum mechanics is equivalent to (an extension of) the well-known geometric quantization of M [7]. It seems however that the representation considered here is of little use in practice, as neither the wave functions nor the observables can be calculated exactly.

The construction depends on the choice of a torsion-free symplectic connection $\nabla = d + \Gamma$ on M , but different choices lead to equivalent quantizations. The explicit form of the operator $\rho(f)$ corresponding to $f \in C^\infty(M)$ is

$$\rho(f) = f + \partial_\mu f y^\mu + \frac{1}{2}(\partial_\mu \partial_\nu - \Gamma_{\mu\nu}^\kappa \partial_\kappa) f y^\mu y^\nu + \mathcal{O}(\hbar^{3/2}), \quad (4.1)$$

and the connection form of the tensor product connection on $\mathcal{H} \otimes B$ is

$$A = -\frac{i}{\hbar}(\theta + \omega_{\mu\nu} y^\mu dx^\nu) - \frac{i}{2\hbar} \Gamma_{\kappa\mu\nu} y^\kappa y^\mu dx^\nu - \frac{i}{8\hbar} R_{\kappa\lambda\mu\nu} y^\kappa y^\lambda y^\mu dx^\nu + \mathcal{O}(\hbar^1), \quad (4.2)$$

where R is the curvature of ∇ ; for details consult [2]. To find the explicit form of the Schrödinger equation (2.13) we need to evaluate A on the velocity vector of a curve satisfying Hamilton's equation $\partial_t c^\mu = \omega^{\mu\nu} \partial_\nu H$. This gives

$$-i\hbar A(\dot{c}) = -\theta_\mu \omega^{\mu\nu} \partial_\nu H - \partial_\mu H y^\mu + \frac{1}{2} \Gamma_{\mu\nu}^\kappa \partial_\kappa H y^\mu y^\nu - \frac{1}{8} R_{\kappa\lambda\mu\nu} \omega^{\nu\alpha} \partial_\alpha H y^\kappa y^\lambda y^\mu + \mathcal{O}(\hbar^2). \quad (4.3)$$

The Schrödinger equation along c , $i\hbar \partial_t \phi = (\rho(H) - i\hbar A(\dot{c}))\phi$, becomes

$$i\hbar \partial_t \phi = \left(H - \theta_\mu \omega^{\mu\nu} \partial_\nu H + \frac{1}{2} \partial_\mu \partial_\nu H y^\mu y^\nu + \mathcal{O}(\hbar^{3/2}) \right) \phi, \quad (4.4)$$

looking exactly as in the flat case up this order in \hbar . Therefore the \hbar -degree of the expectation values $\langle y^{\mu_1} \dots y^{\mu_k} \rangle$ is again not lowered under time evolution along c , and theorem 2.1 remains valid.

One should note however that the quantization method explained in this section is not rigorous. The connection form on $\mathcal{H} \otimes B$ is an infinite power series in \hbar and the canonical operators y^μ , and there seems to be little hope to prove convergence in any sensible way. In certain cases it is possible however to prove that the induced formal star-product on a subalgebra of the observables involves only finitely many \hbar -powers [1, 9], and that it has a well-defined representation which is at least formally equivalent to the one indicated here [7].

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